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LETTER TO THE EDITOR

(2 + 1)-dimensional models with Virasoro-type symmetry algebra

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Abstract. According to a theory proposed in a previous paper by Lou and Hu, starting from a realization of a Virasoro-type symmetry algebra, we can obtain various (2 + 1)-dimensional integrable models under the condition that the models possess the infinitely dimensional Virasoro-type symmetry algebra. An explicit realization of the generalized Virasoro-type symmetry algebra is used to obtain concrete invariant models. A set of equations which possesses the same infinite dimensional Kac–Moody–Virasoro type Lie point symmetry algebra as that of the Kadomtsev–Petviashvilli equation is given.

The integrable and soliton systems in higher-dimensional spacetime have attracted much attention from physicists and mathematicians, e.g. the self-dual Yang–Mills equations have been generalized to $(2n+1)$-dimensional spacetime [1]; the generalized AKNS system has been studied in $\mathbb{R}^{n+1}$ [2]. The symmetry structures of higher-dimensional integrable models have also been widely studied [3–8]. In particular, recent studies [6–8] show us that the generalized higher-order symmetries of some higher-dimensional integrable models, such as the Kadomtsev–Petviashvilli (KP) equation and Toda field equation [7], constitute the generalizations of physically significant W_∞ algebras [9]. However, the usual infinite-dimensional Lie point symmetries may be more important for higher-dimensional models [6, 8]. For some types of (2 + 1)-dimensional integrable models such as the Davey–Stewartson equation [8, 10, 11] and the (2 + 1)-dimensional Sawada–Kotera equation [12], there is no generalized W_∞ algebra constituted by generalized higher-order symmetries with arbitrary functions [8, 12], but they do possess infinite-dimensional Kac–Moody–Virasoro-type Lie point symmetry algebras [5, 11, 12]. It is interesting that among these full Lie point symmetry algebras, they possess a common centreless Virasoro-type subalgebra

$$[\sigma(f_1), \sigma(f_2)] = \sigma(\dot{f}_1 f_2 - \dot{f}_2 f_1) \tag{1}$$

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where f is an arbitrary function of a single independent variable, say time t. It is necessary to point out that the model equations are f-independent. When we take $f(t)$ as $\exp rt$ or t^r ($r = 0, \pm 1, \pm 2, \ldots$) the algebra (1) reduces back to the usual centreless Virasoro algebra. It is our understanding that all the known $(2+1)$-dimensional integrable models, for example the models in [5], have such a symmetry subalgebra and that there is, as yet, no known non-integrable models possessing such a type of symmetry algebra. According to these facts, Lou and Hu proposed a theory [8]: if there exists a symmetry $\sigma(f)$ with arbitrary function $f(t)$ for a high-dimensional model such that $\sigma(f)$ satisfies the algebra (1), then the model may be integrable. Now the problem is how many models are there and how are these integrable models to be obtained under this condition. It is obvious that there exist infinitely many realizations of the algebra (1). Using every realization of algebra (1), there exist infinitely many corresponding group invariants. That is to say, we can obtain infinitely many integrable models under this condition for every realization of algebra (1). In this short letter, for a given realization we would like to describe the general process for obtaining generalized $(2+1)$-dimensional integrable models under the condition that the models possess the Virasoro-type algebra (1). Some well known $(2+1)$-dimensional integrable models will be described in this way. The first step in the derivation of invariant equations is to realize the Lie algebra of the assumed symmetry group in terms of vector fields on the space $X \otimes U$ of independent and dependent variables [1, 13]. In this letter, we restrict X to $(2+1)$-dimensional spacetime with coordinates x, y, t and U is the space of real scalar functions $u(x, y, t)$. The vector fields will all have the form

$$V = X(x, y, t, u)\partial_x + Y(x, y, t, u)\partial_y + T(x, y, t, u)\partial_t + U(x, y, t, u)\partial_u. \tag{2}$$

Without loss of generality, to realize the algebra (1) we can choose f to be an arbitrary function of t and then

$$T(x, y, t, u) = -f(t) \tag{3}$$

while X, Y and U should have the form

$$(X, Y, U) = \sum_{i=1}^{n} f^{(i)}(X_i, Y_i, U_i) \qquad (n = 1, 2, 3, \ldots) \tag{4}$$

where $f^{(i)} = (\mathrm{d}^i/\mathrm{d}t^i)f$ and X_i, Y_i and U_i are functions of (x, y, t, u) and should be selected such that $\sigma = V$ satisfies the algebra (1). In order to obtain some concrete significant results, we restrict n such that $n = 3$ and the vector V possesses the form

$$V_1 = -f(t)\partial_t - (c_2 x\dot{f} + c_5 y^2 \ddot{f})\partial_x - c_3 \dot{f} y \partial_y + (c_1 \dot{f} u + c_4 x \ddot{f} + c_6 y^2 \dddot{f})\partial_u \tag{5}$$

where the dots denote the differentiations with respect to time t, $c_2 = 1 - c_1$, $c_3 = \frac{1}{2}(1 + c_2)$ and c_1, c_4, c_5 and c_6 are four arbitrary constants. It is easy to verify that V_1 given by equation (5) satisfies equation (1).

In order to construct invariant kth-order partial differential equations (PDEs) we need the kth prolongation of the vector field (5). The general formula for the kth prolongation of a vector field is given, for example, by Olver [3] which has the form

$$\begin{aligned}\mathrm{pr}^{(k)}V &= V + U^x\partial_{u_x} + U^y\partial_{u_y} + U^t\partial_{u_t} + U^{xx}\partial_{u_{xx}} + U^{xy}\partial_{u_{xy}} + U^{yy}\partial_{u_{yy}} + U^{xt}\partial_{u_{xt}} + \cdots \\ &= V + \sum_{1 \leqslant i+j+l \leqslant k} U^{x^i y^j t^l}\partial_{u_{x^i y^j t^l}} \qquad (u_{x^i y^j t^l} \equiv \partial_x^i \partial_y^j \partial_t^l u)\end{aligned} \tag{6}$$

$$U^x = D_x(U - Xu_x - Yu_y - Tu_t) + Xu_{xx} + Yu_{xy} + Tu_{xt} \tag{7}$$

$$U^y = D_y(U - Xu_x - Yu_y - Tu_t) + Xu_{xy} + Yu_{yy} + Tu_{yt} \tag{8}$$

$$U^t = D_t(U - Xu_x - Yu_y - Tu_t) + Xu_{xt} + Yu_{ty} + Tu_{tt} \tag{9}$$

$$U^{x^iy^jt^l} = D_xU^{x^{i-1}y^jt^l} - (D_xX)u_{x^iy^jt^l} - (D_xY)u_{x^{i-1}y^{j+1}t^l} - (D_xT)u_{x^{i-1}y^jt^{l+1}} \tag{10}$$

$$U^{x^iy^jt^l} = D_yU^{x^iy^{j-1}t^l} - (D_yX)u_{x^{i+1}y^{j-1}t^l} - (D_yY)u_{x^iy^jt^l} - (D_yT)u_{x^iy^{j-1}t^{l+1}} \tag{11}$$

$$U^{x^iy^jt^l} = D_tU^{x^iy^jt^{l-1}} - (D_tX)u_{x^{i+1}y^jt^{l-1}} - (D_tY)u_{x^iy^{j+1}t^{l-1}} - (D_tT)u_{x^iy^jt^l} \tag{12}$$

where D_x, D_y and D_t are total derivatives.

Using equations (6)–(12), we can calculate the kth prolongation of the vector V_1 defined by equation (5):

$$\begin{aligned}
\mathrm{pr}^{(k)}V_1 = V_1 &+ (\dot f u_x + c_4\ddot f)\partial_{u_x} + [(c_1+c_3)\dot f u_y + 2c_5 y u_x \ddot f + 2c_6 y \dddot f]\partial_{u_y} \\
&+ [(1+c_1)\dot f u_t + (c_1u + c_2xu_x + c_3u_y)\ddot f + (c_4x + c_5y^2u_x)\dddot f + c_6y^2\ddddot f]\partial_{u_t} \\
&+ \sum_{i=2}^{k}(1+(i-1)c_2)\dot f u_{x^i}\partial_{u_{x^i}} + [(1+3c_3)\dot f u_{xy} + 2c_5 y\ddot f u_{xx}]\partial_{u_{xy}} \\
&+ [(1+2c_3)\dot f u_{xyy} + 2c_5\ddot f(u_{xx} + 2yu_{xxy})]\partial_{u_{xyy}} + [(1+3c_3)\dot f u_{xyyy} \\
&+ 6c_5\ddot f(u_{xxy} + yu_{xxyy})]\partial_{u_{xyyy}} + [(1+4c_3)\dot f u_{xy^4} + c_5\ddot f(12u_{xxyy} + 8yu_{xxy^3})]\partial_{u_{xy^4}} \\
&+ [2\dot f u_{yy} + 2c_5\ddot f(u_x + 2yu_{xy}) + 2c_6\dddot f]\partial_{u_{yy}} \\
&+ [(2+c_3)\dot f u_{yyy} + 6c_5\ddot f(u_{xy} + yu_{xyy})]\partial_{u_{yyy}} \\
&+ [(1+c_1+c_3)\dot f u_{ty} + \ddot f((c_1+c_3)u_y + c_2xu_{xy} + c_3yu_{yy} + 2c_5yu_{tx}) \\
&+ c_5\dddot f(2yu_x + y^2u_{xy}) + 2c_6y\ddddot f]\partial_{u_{ty}} \\
&+ [2\dot f u_{tx} + \ddot f(u_x + c_2xu_{xx} + c_3yu_{xy}) + \dddot f(c_4 + c_5y^2u_{xx})]\partial_{u_{tx}} + \cdots .
\end{aligned} \tag{13}$$

For a given integer k, all the other extensions $U^{x^iy^jt^l}$ in (6) for V_1 can be added into equation (13) using recursion relations (10)–(12). We do not give them because of their complexity.

The next procedure to obtain the invariant equations is standard. The general kth-order invariant equation will have the form

$$F(x, y, t, u, u_x, u_y, u_t, \ldots, u_{x^iy^jt^l}, \ldots) = 0 \qquad (0 \leqslant i + j + l \leqslant k) \tag{14}$$

where the function F satisfies

$$\mathrm{pr}^{(k)}V_1 \cdot F = 0. \tag{15}$$

Thus all we have to do is find the characteristics for equation (15) in which all the arguments in (14) are viewed as independent variables. The characteristics will provide us with a set of elementary invariants $I_n(x, y, t, u, \ldots, u_{x^iy^jt^l}, \ldots)(n = 1, 2, \ldots, N)$ and the general invariant equation is an arbitrary function of the elementary invariants I_n:

$$H(I_1, I_2, \ldots, I_N) = 0. \tag{16}$$

Generally, the elementary invariants I_n (and then $H(I_1, I_2, \ldots, I_n)$) are f-dependent. As in the known cases, to obtain new integrable models we should select *f-independent* model equations from equation (16).

To get the explicit elementary invariants of V_1, we have to solve the characteristic equations

$$\frac{\mathrm{d}x}{-c_2xf} = \frac{\mathrm{d}y}{-c_3y\dot{f}} = \frac{\mathrm{d}t}{-f} = \frac{\mathrm{d}u}{-c_1u\dot{f} + c_4x\ddot{f} + c_6y^2\dddot{f}} = \frac{\mathrm{d}u_x}{u_x\dot{f} + c_4\ddot{f}} = \cdots$$
$$= \frac{\mathrm{d}u_{x^iy^jt^l}}{U^{x^iy^jt^l}} = \cdots \tag{17}$$

where some of the $U^{x^iy^jt^l}$ can be read off from (13) directly, while others can be obtained from recursion relations (10)–(12). After finishing some detailed calculations of equation (17), we can obtain various group invariants. Here are some examples:

$$J_n = u_{nx}f^{1+(n-1)c_2} \qquad (u_{nx} \equiv u_{x^n} \equiv \partial_x^n u \qquad n = 2, 3, 4, \ldots, k) \tag{18}$$

$$I_1 = yf^{-c_3} \tag{19}$$

$$I_2 = xf^{-c_2} - c_5y^2f^{-2c_3}\dot{f} \tag{20}$$

$$I_3 = uf^{c_1} + c_4xf^{-c_2}\dot{f} - \tfrac{1}{2}(c_4c_5 + c_6)y^2f^{-2c_3}\dot{f}^2 + c_6y^2f^{1-2c_3}\ddot{f} \tag{21}$$

$$I_4 = u_xf + c_4\dot{f} \tag{22}$$

$$I_5 = u_yf^{c_1+c_3} + c_5yf^{-c_3}\dot{f}(2u_xf + c_4\dot{f}) + 2c_6yf^{1-c_3}\ddot{f} - c_6yf^{-c_3}\dot{f}^2 \tag{23}$$

$$I_6 = u_tf^{c_1+1} + (c_1I_3 + c_2I_2I_4 + c_3I_1I_5)\dot{f}) + (c_4I_2 + c_5I_1^2I_4)(f\ddot{f} - \dot{f}^2)$$
$$+ c_6y^2f^{-2c_3}(f^2\dddot{f} + \dot{f}^3 - 2f\dot{f}\ddot{f}) \tag{24}$$

$$I_7 = u_{xy}f^{c_3+1} + 2c_5yu_{xx}f^{c_3}\dot{f} \tag{25}$$

$$I_8 = u_{xxy}f^{c_2+c_3+1} + 2c_5yu_{xxx}f^{c_3+c_2}\dot{f} \tag{26}$$

$$I_9 = u_{yy}f^2 + 2c_5(2yu_{xy}f + 2y^2c_5u_{xx}\dot{f} + u_xf)\dot{f} + (c_4c_5 - c_6)\dot{f}^2 + 2c_6f\ddot{f} \tag{27}$$

$$I_{10} = u_{tx}f^2 + [u_xf + c_2xu_{xx}f^{2c_3-c_2} - (c_2-2c_3+1)c_5y^2u_{xx}\dot{f} + c_3yu_{xy}f]\dot{f} + (c_4+c_5y^2u_{xx})f\ddot{f} \tag{28}$$

$$I_{11} = u_{xyy}f^{2c_3+1} + 2c_5(J_2 + 2I_1I_8)\dot{f} - 4c_5^2I_1^2J_3\dot{f}^2 \tag{29}$$

$$I_{12} = u_{yyy}f^{c_3+2} + 6c_5(I_7 + I_1I_{11})\dot{f} - 12c_5^2(I_1J_2 + I_1^2I_8)\dot{f}^2 + 8c_5^3I_1^3J_3\dot{f}^3 \tag{30}$$

$$I_{13} = u_{txx}f^{c_2+2} + (2c_3J_2 + c_2I_2J_3 + c_3I_1I_8)\dot{f} + c_5I_1^2J_3(f\ddot{f} - \dot{f}^2) \tag{31}$$

$$I_{14} = u_{txy}f^{c_3+2} + [(c_3 + 1)I_7 + c_2I_2I_8 + c_3I_1I_{11} + 2c_5I_1I_{13}]\dot{f} + (2c_5I_1J_2 + c_5I_1^2I_8)f\ddot{f}$$

$$- [(2c_2+4)c_5 I_1 J_2 + 2c_2 c_5 I_1 I_2 J_3 + (c_2+2)c_5 I_1^2 I_8]\dot{f}^2$$
$$- 2c_5^2 I_1^3 J_3 f \dot{f} \ddot{f} + 2c_5^2 I_1^3 J_3 \dot{f}^3 \qquad (32)$$

$$I_{15} = u_{xxxy} f^{2c_2+c_3+1} + 2c_5 I_1 J_4 \dot{f} \qquad (33)$$

$$I_{16} = u_{xxyy} f^{2c_2+2} + 2c_5(J_3 + 2I_1 I_{15})\dot{f} - 4c_5^2 I_1^2 J_4 \dot{f}^2 \qquad (34)$$

$$I_{17} = u_{xyyy} f^{3c_3+1} + 6c_5(I_8 + I_1 I_{16})\dot{f} - 12c_5^2(I_1 J_3 + I_1^2 I_{15})\dot{f}^2 + 8c_5^3 I_1^3 J_4 \dot{f}^3 \qquad (35)$$

$$I_{18} = u_{yyyy} f^{2c_3+2} + 4c_5(3I_{11} + 2I_1 I_{17})\dot{f} - 12c_5^2(J_2 + 4I_1 I_8 + 2I_1^2 I_{16})\dot{f}^2$$
$$+ 16c_5^3(3I_1^2 J_3 + 2I_1^3 I_{15})\dot{f}^3 - 16c_5^4 I_1^4 J_4 \dot{f}^4 \qquad (36)$$

$$I_{19} = u_{ty} f^{c_1+c_3+1} + [(c_1+c_3)I_5 + c_2 I_2 I_7 + c_3 I_1 I_9 + 2c_5 I_1 I_{10}]\dot{f}$$
$$- c_5 I_1[4I_4 + (1+2c_3)I_1 I_7 + 2c_2 I_2 J_2]\dot{f}^2 + c_5 I_1(2I_4 + I_1 I_7) f \ddot{f} + 2c_6 I_1 f^2 \dddot{f}$$
$$- 2I_1(2c_6 + c_4 c_5 + c_5^2 I_1^2 J_2) f \dot{f} \ddot{f} + 2I_1(c_6 + c_4 c_5 + c_5^2 I_1^2 J_2)\dot{f}^3. \qquad (37)$$

Hence the most general V_1 invariant equation is

$$H(J_n, I_m, n = 2, 4, 3, \ldots, k; m = 1, 2, 3, \ldots) = 0. \qquad (38)$$

Generally, the models obtained from equation (3) are quite complicated and f-dependent. In order to obtain integrable models we should select out the models which are *f-independent* from equation (38); this is still a difficult task because the elementary invariants are dependent on the arbitrary function f in a complicated manner. Fortunately, some physically significant models can be read from equation (38). For instance, if we fix the constants c_i, $i = 1, 2, \ldots, 6$, as

$$c_1 = \frac{2}{3} \qquad c_2 = \frac{1}{3} \qquad c_3 = \frac{2}{3} \qquad c_4 = -\frac{1}{3A} \qquad c_5 = -\frac{1}{6B} \qquad c_6 = \frac{1}{6AB} \qquad (39)$$

with three arbitrary constants A, B and C, the following group invariant equation

$$H_1 \equiv I_{10} + AI_3 J_2 + AI_4^2 + BI_9 + CJ_4 = 0 \qquad (40)$$

is really an *f-independent* integrable evolution equation:

$$u_{tx} + (Auu_x + Cu_{xxx})_x + Bu_{yy} = 0. \qquad (41)$$

Equation (41) is just the well known KP equation for $C \neq 0$ and the Khokhlov–Zabolotskaya (KZ) equation [14] for $C = 0$. Of course the KP equation and the KZ equation are integrable under the other traditional condition. From equation (38) we can obtain infinitely many *f-independent* invariant equations; here are two other examples:

$$u_{xx}[u_{tx} + (Auu_x + Cu_{xxx})_x + Bu_{yy}] + Eu_{xxx}^2 = 0 \qquad (42)$$

and

$$u_{tx} + Auu_{xx} + Au_x^2 + Bu_{yy} + \sum_{n=2}^{M} C_n u_{nx}^{6/(n+2)} = 0 \qquad (M = 2, 3, \ldots) \qquad (43)$$

with the same condition (39) while A, B, C_n ($C_4 \equiv C$) and E are arbitrary constants. Equations (42) and (43) can be read from

$$J_2 H_1 + E J_3^2 = 0 \tag{44}$$

and

$$H_1 + \sum_{n=2}^{M} C_n J_n^{6/(n+2)} = 0 \qquad (M = 2, 3, \ldots) \tag{45}$$

respectively. Using the formal series symmetry approach [7] and/or the standard Lie point symmetry approach, one can easily verify that the models (43) and (44) possess exactly the same Lie point symmetries as those of the KP equation which have been given in [15] and constitute the Kac–Moody–Virasoro-type algebra structure.

Using different realizations of the Virasoro-type algebra (1) and the same procedure given here, not only can we obtain all the known (2 + 1)-dimensional integrable models but also a large number of new integrable models under the condition that they possess the Virasoro-type infinite dimensional symmetry algebra. Models obtained in this way need to be studied further, in particular to find whether the new models are integrable or not under the other traditional condition.

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